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On the stabilization of approximation procedures for a class of integral equations of the first kind with noisy right-hand sides

G. BRUCKNER*

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Abstract — An interpolation projection method is proposed to stabilize an arbitrary given numerical procedure for solution of a certain class of first kind linear integral equations in the case of inexact right-hand sides. The proposed interpolation method can be chosen adequate to pointwise given measured values and an *a priori* information on the solution. In the one-dimensional periodic case examples of interpolation are given using trigonometric polynomials, splines or multiresolution spaces. The resulting convergence rates in dependence on the noise level are optimal in the usual sense of inverse theory.

1. INTRODUCTION

Many boundary value problems can be formulated by various familiar methods of potential theory as boundary integral equations of the first kind

$$Au = g \tag{1.1}$$

with a smoothing operator A. Some of the easiest examples of such integral equations are the following:

$$\begin{aligned}
\frac{1}{2\pi}\int_\Gamma u(y)\log\frac{1}{|x-y|}\,\mathrm{d}\gamma_y &= f(x)\\
\frac{1}{4\pi}\int_\Omega \frac{U(g)}{|x-y|}\,\mathrm{d}\omega_y &= F(x)\\
\frac{\mathrm{i}}{4}\int_\Gamma H_0^{(1)}(k|x-y|)\,v(y)\,\mathrm{d}\gamma_y &= g(x)\\
\frac{1}{4\pi}\int_\Omega \frac{\mathrm{e}^{-\mathrm{i}k|x-y|}}{|x-y|}V(y)\,\mathrm{d}\omega_y &= G(x)
\end{aligned}$$

where $\Gamma \subset \mathbb{R}^2$ is a smooth curve and $\Omega \subset \mathbb{R}^3$ is a smooth area. These examples correspond to the Dirichlet problem for the Laplace and Helmholtz equations in the two-dimensional

*Weierstrass Institute for Applied Analysis and Stochastics, Mohrenstraße 39, D–10117 Berlin, Germany. E-mail: bruckner@wias-berlin.de

and three-dimensional cases respectively (cf., e. g., [4, Chapter XI], [10]). Such integral equations have been investigated very well with respect to mapping properties of the operator, the well-posedness in suitable spaces and numerical methods in the case of exactly given data (cf., e. g., [3, 10, 13, 14, 16] and references therein).

In this paper we are engaged with noisy right-hand sides where the error is supposed to have a magnitude of about 0.1 to 1 percent, and one has to develop regularization methods for stabilization. For some of the problems special kinds of regularization procedures were considered, e. g., in [7–9].

In continuation of [2] the aim of this paper is to develop a general universally applicable regularization method by decomposing the problem (1.1) into a well-posed part and an ill-posed part. For short we will call the well-posed part, depending on the actual point of view, 'Part I' or 'Problem I' or 'Procedure I', and the ill-posed part 'Part II' or 'Problem II' or 'Procedure II', respectively.

In this paper Part I where the problem (1.1) is solved with an exactly given right-hand side is considered as known. We will be engaged only with Part II where the disturbed right-hand side and the range space of the operator A are entering.

Part II was solved in [2] by regularizing the imbedding of the state space, i. e., the range of A, into the observation space, i. e., the data space, by truncated singular value decomposition. Here we consider an approximation procedure for g from pointwise given noisy data. In the sense of a regularization by discretization (cf. [15]) it can be interpreted as a collocation method where the places of measurements are the collocation points.

The approximation problem is solvable if the trial spaces possess the following three properties: the approximation property, the inverse property, and the (so-called) finite property. The latter says that the norm of a trial function can be estimated by a finite norm taken at the collocation points.

The approach by decomposition is advantageous as Procedures I and II can be combined nearly without additional conditions. One has only to be sure that the approximation takes place in the range of A. Moreover, for the approximation one has a free choice of the trial spaces. They can be adapted to the *a priori* information on the solution. Therefore the decomposition method offers the possibility of a twofold adaption: Part I can be chosen in accordance with the operator, while Part II can be tuned to the right-hand side and the data. These are similar advantages as in the wavelet-vaguelette decomposition method (cf. [5, 12]).

If the noise level decreases to zero, then the rate of convergence will be equal in both Procedure II and the whole problem. This is not surprising since Part II represents the ill-posed part of the whole problem. Here Part II can be considered as a projection procedure that is regularized by a proper choice of the number of equidistant places of measurement. The rate of convergence will be optimal in the usual sense of optimality for linear regularization methods (cf., e. g., [11]).

In Section 2 the method will be explained, and in Section 3 some examples of trial spaces are given possessing the three properties mentioned above.

2. THE METHOD

Let us start with an abstract formulation of the problem and its Parts I and II. The integral equations of the first kind mentioned in the introduction can be described in the following way.

Let X and Y be Banach spaces, and let A be a linear continuous operator mapping

X isomorphically onto Y. Then we seek for the (unique) solution u of the equation

$$Au = g \tag{2.1}$$

where $g \in Y$ is given unexactly in some sense to be made precise later.

We will say that Problem I is solved when for every (exactly given) $g \in Y$ an approximation $T_h g$ can be computed. Here $h > 0$ is a discretization parameter; $T_h : Y \to X_h$ is a linear continuous operator, $h > 0$; $X_h \subset X$ is a finite-dimensional subspace, $h > 0$; and for every $g \in Y$ we have

$$\|A^{-1}g - T_h g\|_X \to 0 \quad (h \to 0). \tag{2.2}$$

From (2.2) we obtain the stability assertion

$$\|T_h\| \leq c. \tag{2.3}$$

To formulate Problem II let us suppose additionally that the right-hand side g is defined as a function on the domain Ω and is given on a finite mesh

$$G_n = \{t_j\}_1^n, \quad t_j \in \Omega$$

by inaccurate values g_j^ε where

$$|g(t_j) - g_j^\varepsilon| \leq \varepsilon, \quad j = 1, \ldots, n. \tag{2.4}$$

Here $\varepsilon > 0$ is a given noise level, $n \in \mathbb{N}$ is the number of mesh points. Let further $d > 0$ be a discretization parameter (e. g., the maximal length in a triangulation of Ω). It is clear that d is related to the number of mesh points n. Now Problem II consists in finding elements

$$P(d, \varepsilon) \in Y \tag{2.5}$$

and a parameter $d = d(\varepsilon)$ such that

$$\|g - P(d, \varepsilon)\|_Y \to 0 \quad (\varepsilon \to 0) \tag{2.6}$$

where the rate of convergence is as high as possible.

As solution of the whole problem (2.1) where the right-hand side is given inaccurately in the sense of (2.4) we will consider the method

$$T_h P(d, \varepsilon), \quad d = d(\varepsilon), \quad h = h(\varepsilon) \tag{2.7}$$

whose convergence properties because of (2.3) are characterized by

$$\|A^{-1}g - T_h P(d, \varepsilon)\|_X \leq c\,\|g - P(d, \varepsilon)\|_Y + \|A^{-1}g - T_h g\|_X. \tag{2.8}$$

This means that the ill-posedness of the whole problem is contained in the approximation problem (2.6) (i. e., in Part II). Here d is the regularization parameter. The discretization parameter $h = h(\varepsilon)$ can be arbitrarily small, however from reasons of effectiveness it should be such that both norms on the right-hand side of (2.8) are of the same magnitude.

For the rest of the paper we will consider mainly Problem II. This is an approximation problem for the right-hand side g of (2.1). Since the measured values are given at the discrete mesh points t_j, $j = 1, \ldots, n$, we will consider a collocation procedure whose solution is an interpolation of the measured values. Then regularization means that the accuracy and number of measured values, i. e., the collocation points, are in a balanced relation to each other.

In the sequel, for the sake of simplicity let us characterize the regularization parameter d by the number of measured values, i. e., the number of mesh points n.

Let Ω be a bounded domain (or a manifold), and let Y, Z be Banach spaces of complex-valued functions on Ω. Further, let $Y \subset C(\Omega)$, and let

$$Y = Y^{\lambda_0} \tag{2.9}$$

be a member of a scale of spaces Y^λ, $\lambda \geq \lambda_0$. Let Y be densely and continuously imbedded in Z,

$$Y \subset Z \tag{2.10}$$

and let

$$Y_n \subset Y^\lambda, \quad \lambda \geq \lambda_0, \quad Y_n \subset Y_{n+1}, \quad n \in \mathbb{N}_0 = \{0, 1, \ldots\} \tag{2.11}$$

be hierarchically ordered finite-dimensional subspaces. Further, let a mesh

$$G_n = \{t_j, \ldots, t_n\}, \quad t_j \in \Omega, \quad n \in \mathbb{N} \tag{2.12}$$

be given on Ω.

Additionally, for arbitrary $\underline{g} = (g_1, \ldots, g_n)$, $g_j \in \mathbb{C}$, $j = 1, \ldots, n$, let there exist a uniquely determined interpolation polynomial

$$S_n \underline{g} \in Y_n \tag{2.13}$$

with the property

$$(S_n \underline{g})(t_j) = g_j, \quad j = 1, \ldots, n. \tag{2.14}$$

For $g \in C(\Omega)$, $\underline{g} = (g(t_1), \ldots, g(t_n))$, let us define

$$S_n g = S_n \underline{g}.$$

The operator S_n defined above is a projector of Y onto Y_n.

For the operator S_n and the subspaces Y_n we consider the following properties:

Approximation property. If $g \in Y^\lambda$, $\lambda \geq \lambda_0$, we have

$$\|g - S_n g\|_Y \leq c n^{\lambda_0 - \lambda} \|g\|_{Y^\lambda}. \tag{2.15}$$

Inverse property. There exists $a \geq 0$ such that for all $\psi \in Y_n$ we have

$$\|\psi\|_Y \leq c \cdot n^a \|\psi\|_Z. \tag{2.16}$$

Finite property. For all $\psi \in Y_n$ we have

$$\|\psi\|_Z \leq c \max_{1 \leq j \leq n} |\psi(t_j)|. \tag{2.17}$$

Here c is a generic constant that can be allowed to depend on n (cf. Example 3.2 in Section 3).

Now let the vector of measured values

$$\underline{g}^\varepsilon = (g_1^\varepsilon, \ldots, g_n^\varepsilon)$$

be given with the property (2.4) for the right-hand side g. The following statement holds:

Theorem 2.1. *Let the operator S_n and the spaces Y_n possess the properties* (2.15)–(2.17). *Then for $g \in Y^\lambda$ we obtain*

$$\|g - S_n\underline{g}^\varepsilon\|_Y \le c\left(n^{\lambda_0-\lambda}\|g\|_{Y^\lambda} + n^a\varepsilon\right). \tag{2.18}$$

If

$$n \sim \varepsilon^{-\frac{1}{\lambda-\lambda_0+a}} \tag{2.19}$$

then we have

$$\|g - S_n\underline{g}^\varepsilon\|_Y = O\left(\varepsilon^{\frac{\lambda-\lambda_0}{\lambda-\lambda_0+a}}\right). \tag{2.20}$$

Proof. From (2.15)–(2.17) we get

$$\begin{aligned}\|g - S_n\underline{g}^\varepsilon\|_Y &\le \|g - S_n g\|_Y + \|S_n g - S_n\underline{g}^\varepsilon\|_Y \\ &\le cn^{\lambda_0-\lambda}\|g\|_{Y^\lambda} + cn^a\|S_n g - S_n\underline{g}^\varepsilon\|_Z \\ &\le c\left\{n^{\lambda_0-\lambda}\|g\|_{Y^\lambda} + n^a \max_{1\le j\le n}|g(t_j) - g_j^\varepsilon|\right\}\end{aligned}$$

since $S_n g - S_n\underline{g}^\varepsilon = S_n(g - \underline{g}^\varepsilon) \in Y_n$ and because of the properties (2.14) and (2.4). □

The rate of convergence (2.20) is optimal as can be seen for the special case when $\Omega = \mathbb{R}/\mathbb{Z}$, $Y^\lambda = H^\lambda(\Omega)$, $Y = Y^1$, $Z = L_2(\Omega)$, and S_n is the operator of trigonometric interpolation. Here $\lambda_0 = 1$, $a = 1$, and for $g \in H^\lambda(\Omega)$, $\lambda > 1$, we obtain

$$\|g - S_n\underline{g}^\varepsilon\|_Y = O\left(\varepsilon^{\frac{\lambda-1}{\lambda}}\right).$$

This is the optimal rate for linear regularization procedures in this case. It is clear from (2.8) for

$$P(d,\varepsilon) = S_n\underline{g}^\varepsilon$$

where in the just considered special case $d = n^{-1}$ can be taken for an equidistant mesh, that if

$$h = h(\varepsilon) \tag{2.21}$$

is small enough, then (2.20) is also the rate of convergence of the procedure

$$T_h S_n\underline{g}^\varepsilon \tag{2.22}$$

for the problem (2.1). The choice (2.21) then depends on the rate of the procedure T_h and has to be taken such that $\|A^{-1}g - T_h g\|_X$ behaves like $\|g - S_n\underline{g}^\varepsilon\|_Y$ $(\varepsilon \to 0)$. The numerical cost of the procedure (2.22) is just the cost of the procedure T_h increased by the construction of the interpolant $S_n\underline{g}^\varepsilon$ from the measured data, where n and ε are related like (2.19).

The explained approximation method for the right-hand side of (2.1) can be interpreted as a projection method regularizing the imbedding $Y \subset Z$. We want to relate the properties (2.15)–(2.17) to the notions of quasioptimality and robustness used in [15].

To this end let us consider the case $Z = C(\Omega)$ and data $g^\varepsilon \in C(\Omega)$ where $\|g^\varepsilon - g\|_C \leq \varepsilon$. The trial spaces Y_n are free to our disposal while the test spaces $Z_n' \subset C^*(\Omega)$ are taken as

$$Z_n' = \operatorname{span}\{\delta_j,\ j = 1, \ldots, n\}, \quad \delta_j g = g(t_j), \quad j = 1, \ldots, n.$$

The operators P_n and Q_n considered in [15] are here the operators S_n taken as projectors of Y onto Y_n and of $C(\Omega)$ onto Y_n respectively. The property

$$S_n g \to g \quad (n \to \infty)$$

for each $g \in Y$ is related to the property (2.15) and, by the Banach–Steinhaus theorem, implies $\|P_n\| \leq c$, i. e., the quasioptimality. For the quantities

$$\alpha_n = \sup\Big\{\|\psi\|_Y,\ \psi \in Y_n,\ \|\psi\|_C = 1\Big\}$$

we obtain from (2.16) the estimate

$$\alpha_n \leq c n^a.$$

Finally, let us show that the property (2.17) implies the robustness. Let g be an arbitrary element of $C(\Omega)$. Then

$$\|Q_n g\|_Y = \|S_n g\|_Y = \frac{\|S_n g\|_Y}{\|S_n g\|_C} \|S_n g\|_C \leq \alpha_n \|S_n g\|_C.$$

Moreover, from (2.17)

$$\|S_n g\|_C \leq c \max_{1 \leq j \leq n} |g(t_j)| \leq c\,\|g\|_C$$

such that

$$\|Q_n g\|_Y \leq c \cdot \alpha_n \|g\|_C$$

i. e., $\|Q_n\| \leq c \cdot \alpha_n$. This is robustness.

3. EXAMPLES

In this section we will give some examples for constellations

$$\Omega,\ Y^\lambda,\ Y,\ Z,\ G_n,\ Y_n,\ S_n$$

where the properties (2.15)–(2.17) take place.

Example 3.1. Trigonometric interpolation in Sobolev spaces (dimension 1). Let us choose

$$\Omega = \mathbb{T} \qquad \text{(the one-dimensional torus } \mathbb{R}/\mathbb{Z});$$

$$Y^\lambda = H^\lambda(\Omega) \qquad \text{(the Sobolev spaces of } [0,1]\text{-periodic functions, norm } \|\cdot\|_\lambda);$$

$$Y = H^{\lambda_0}(\Omega), \quad 0 \leq \lambda_0 \leq \lambda;$$

$$Z = L_2(\Omega) = H^0(\Omega);$$

$$Y_n = \operatorname{span}\Big\{\varphi_l,\ -n/2 < l \leq n/2\Big\}, \quad \varphi_l = e^{2\pi i l x}, \quad n \in \mathbb{N}$$

(the spaces of trigonometric polynomials; we have $Y_n \subset H^s$, $n \in \mathbb{N}$, $s \in \mathbb{R}$);

$$G_n = \left(0, \frac{1}{n}, \ldots, \frac{n-1}{n}\right) \qquad \text{(equidistant mesh on } \mathbb{T});$$

and let S_n be the operator of trigonometric interpolation. Then the following assertions hold true.

1. If $g \in H^\lambda$, $\lambda > 1/2$, $0 \le \lambda_0 \le \lambda$, we have

$$\|g - S_n g\|_{\lambda_0} \le c \cdot n^{\lambda_0 - \lambda} \|g\|_\lambda$$

(approximation property; cf. [1] or [16]).

2. If $r \ge t$, $r, t \in \mathbb{R}$, then for all $\psi \in Y_n$

$$\|\psi\|_r \le c \cdot n^{r-t} \|\psi\|_t$$

(inverse property; in particular, $\|\psi\|_Y \le c \cdot n^{\lambda_0} \|\psi\|_Z$ is true).

3. For arbitrary $\psi \in Y_n$ we have

$$\|\psi\|_Z \le \max_{1 \le j \le n} |\psi(t_j)|, \quad t_j \in G_n$$

(finite property); it follows from the known relation

$$\|\psi\|_0 = \left(\frac{1}{n} \sum_{1 \le j \le n} |\psi(j/n)|^2 \right)^{1/2}.$$

Example 3.2. Trigonometric interpolation in Hölder – Zygmund spaces (dimension 1). Here we choose

$$\begin{aligned}
\Omega &= \mathbb{T} && \text{(as in Example 3.1);} \\
Y^\lambda &= \mathcal{H}^\lambda(\Omega) && \text{(the Hölder – Zygmund spaces of } [0,1]\text{-periodic functions} \\
& && \quad \text{with the norm } \|\cdot\|_{\mathcal{H}^\lambda}); \\
Y &= \mathcal{H}^{\lambda_0}(\Omega), && 0 \le \lambda_0 \le \lambda; \\
Z &= C(\Omega);
\end{aligned}$$

and Y_n, G_n, S_n as in Example 3.1. Again we have $Y_n \subset \mathcal{H}^s$, $n \in \mathbb{N}$, $s \in \mathbb{R}$. Here the following assertions hold true.

1. If $g \in \mathcal{H}^\lambda$, $0 \le \lambda_0 \le \lambda$, then

$$\|g - S_n g\|_{\mathcal{H}^{\lambda_0}} \le c \cdot (\log n)\, n^{\lambda_0 - \lambda} \|g\|_{\mathcal{H}^\lambda}$$

(approximation property, cf. [13, 14, 16]).

2. For $r \ge t$, $r, t \in \mathbb{R}$, we have for all $\psi \in Y_n$

$$\|\psi\|_{\mathcal{H}^r} \le c \cdot n^{r-t} \|\psi\|_{\mathcal{H}^t}$$

(inverse property; in particular, $\|\psi\|_Y \le c \cdot n^{\lambda_0} \|\psi\|_Z$ holds).

3. For arbitrary $\psi \in Y_n$ we have

$$\|\psi\|_C \le c\,(\log n) \max_{1 \le j \le n} |\psi(t_j)|, \quad t_j \in G_n$$

(finite property). This property can be proved as follows. Given $\psi \in Y_n$, let φ be the linear interpolant of the values $\psi(t_j)$, $j = 1, \ldots, n$. Then $\varphi \in C(\Omega)$, $\varphi(t_j) = \psi(t_j)$, and $\|\varphi\|_C \le \max_{1 \le j \le n} |\varphi(t_j)|$ holds. Thus we have

$$\|\psi\|_C = \|S_n \varphi\|_C \le c\,(\log n) \|\varphi\|_C \le c\,(\log n) \max_{1 \le j \le n} |\psi(t_j)|.$$

Note that the approximation and finite properties in Example 3.2 as compared to (2.15) and (2.17) contain the function $\log n$. Consequently, (2.18) reads here as

$$\|g - S_n \underline{g}^\varepsilon\|_Y \le c \cdot \log n \cdot \left(n^{\lambda_0 - \lambda} \|g\|_{Y^\lambda} + n^a \varepsilon\right)$$

and the rate (2.20) can be written as

$$\|g - S_n \underline{g}^\varepsilon\|_Y = O\left(|\log \varepsilon|\, \varepsilon^{\frac{\lambda - \lambda_0}{\lambda - \lambda_0 + a}}\right).$$

Example 3.3. Spline interpolation in Sobolev spaces (dimension 1). We choose

$\Omega, Y^\lambda, Y, Z, G_n$ as in Example 3.1;

$Y_n = \Sigma_d^n$ (the space of $(d-1)$ times continuously differentiable splines of degree d interpolating given values at $t_j \in G_n$, $j = 1, \dots, n$; for example, for $d = 1$ it is the space of linear interpolants);

S_n is the operator of spline interpolation.

Here $\Sigma_d^n \subset H^s$ holds if and only if $s < d + 1/2$ (cf. [6]). We have the following assertions.

1. If d is odd, $g \in H^\lambda$, $\lambda > 1/2$, $0 \le \lambda_0 \le \lambda \le d + 1$, $\lambda_0 < d + 1/2$, then we have

$$\|g - S_n g\|_{\lambda_0} \le c \cdot n^{\lambda_0 - \lambda} \|g\|_\lambda$$

(approximation property, proof in [6]).

2. If $d + 1/2 > r \ge t$, we obtain for all $\psi \in Y_n$

$$\|\psi\|_r \le c \cdot n^{r-t} \|\psi\|_t$$

(inverse property, proof in [6]; in particular, $\|\psi\|_Y \le c n^{\lambda_0} \|\psi\|_0$ holds for $d + 1/2 > \lambda_0 > 0$).

3. In the case $d = 1$ we obtain for an arbitrary $\psi \in Y_n$

$$\|\psi\|_0 \le c \cdot \max_{1 \le j \le n} |\psi(t_j)|$$

(finite property; from $\|\psi\|_C = \max_{1 \le j \le n} |\psi(t_j)|$).

Example 3.4. Multiscale resolution, interpolation in Sobolev spaces (dimension 1). Here our trial spaces are generated by a refinable function. The restriction to dimension one is not essential. The representation is based on [3]. As in Examples 3.1 and 3.3 we choose

$$\Omega = \mathbb{T}; \qquad Y^\lambda = H^\lambda; \qquad Y = H^{\lambda_0}, \quad 0 \le \lambda_0 \le \lambda; \qquad Z = H^0 = L_2.$$

Let

$$\mathbb{Z}_n = \mathbb{Z}/2^n\mathbb{Z}, \quad n \in \mathbb{N}$$

be the additive group modulo 2^n. It is represented by $0, 1, \dots, 2^n - 1$. For $k \in \mathbb{Z}_n$ and fixed $x_0 \in \mathbb{T}$ let us consider

$$G_n = \left\{t_k^n,\ k = 0, 1, \dots, 2^n - 1\right\}, \quad t_k^n = 2^{-n}(x_0 + k).$$

G_n is an equidistant mesh on $\mathbb{T}$ with the mesh points

$$\frac{x_0}{2^n}, \quad \frac{x_0}{2^n}+\frac{1}{2^n}, \quad \ldots, \quad \frac{x_0}{2^n}+\frac{2^n-1}{2^n}.$$

Now let us prepare the definition of the spaces Y_n. For $x_0 \in \mathbb{T}$ the functional $\eta = \delta(\cdot - x_0)$ is linear and bounded on H^s, $s > 1/2$, with the property

$$\eta(f) = f(x_0), \quad f \in H^s.$$

Defining for $k \in \mathbb{Z}_n$

$$\eta_k^n(f) = 2^{-n/2}\,\eta\Big(f(2^{-n}(\cdot + k))\Big)$$

we get

$$\eta_k^n(f) = 2^{-n/a} f(t_k^n).$$

Let $\varphi \in L_2(\mathbb{R})$ be a function with the following properties (P1)–(P5) (we cite from [3]):

(P1) φ has a compact support;

(P2) φ has (algebraic) linearly independent integer translates $\varphi(\cdot - k)$, $k \in \mathbb{Z}$;

(P3) φ is refinable, i.e., there exists a finitely supported mask $a = (a_k)_{k\in\mathbf{Z}}$ such that $\varphi(\cdot) = \sum_{k\in\mathbf{Z}} a_k \varphi(2\cdot - k)$;

(P4) $\varphi \in C^{d'}(\mathbb{R})$ for some $d' \in \mathbb{N}_0$;

(P5) φ is accurate of degree d, i.e., there exists $d \in \mathbb{N}_0$ such that for every polynomial p of degree $\deg(p) \le d$ one has

$$\sum_{k\in\mathbf{Z}} p(k)\,\varphi(\cdot - k) = p(\cdot) + q(\cdot)$$

where q is some polynomial of degree $\deg(q) < \deg(p)$.

Let us consider for $k \in \mathbb{Z}_n$ the periodizations

$$\varphi_k^n(\cdot) = 2^{n/2} \sum_{k'\in\mathbf{Z}} \varphi(2^n \cdot - k + k').$$

From property (P1) we obtain $\varphi_k^n \in L_2(\mathbb{T})$. Now, we are ready to define

$$Y_n = \operatorname{span}\{\varphi_k^n,\ k \in \mathbb{Z}_n\}.$$

From property (P3) we get

$$Y_0 \subset Y_1 \subset \ldots \subset Y_n \subset Y_{n+1} \subset \ldots, \qquad \overline{\cup_{n\in\mathbb{N}_0} Y_n} = L_2.$$

Further, from property (P4) we have

$$Y_n \subset H^s, \quad s < d' + \varrho$$

where ϱ is such that $0 < \varrho < 1$ and

$$\left|(\partial_x^{d'}\varphi)(x) - (\partial_x^{d'}\varphi)(y)\right| \le c|x-y|^{\varrho}.$$

The following is shown in [3]: From a refinable function φ with properties (P1)–(P5) another refinable function

$$\phi = \sum_{k\in\mathbf{Z}} g_k \varphi(\cdot - k)$$

can be constructed also satisfying (P1)–(P5) but having the additional property

(P6) $\eta_k^n(\phi_{k'}^n) = 2^{-n/2}\phi_{k'}^n(t_k^n) = \delta_{kk'},\ \ k, k' \in \mathbb{Z}_n$.

Therefore, from now on let us suppose that φ possesses properties (P1)–(P6). Let us define for $v \in H^s$, $s > 1/2$,

$$S_n v = 2^{-n/2} \sum_{k\in\mathbb{Z}_n} v(t_k^n)\,\varphi_k^n \quad \Big(= \sum_{k\in\mathbb{Z}_n} \eta_k^n(v)\,\varphi_k^n\Big).$$

S_n is a projector from H^s onto Y_n, the element $S_n v$ only depends on the vector

$$\underline{v} = (v(t_k^n))_{k\in\mathbb{Z}_n}$$

and for every vector

$$\underline{g} = (g_k^n)_{k\in\mathbb{Z}_n}$$

there is a unique element

$$S_n g = \sum_{k\in\mathbb{Z}_n} 2^{-n/2} g_k^n\,\varphi_k^n \in Y_n$$

with the property

$$(S_n\underline{g})(t_k^n) = g_k^n.$$

We have the following assertions:

1. If $g \in H^\lambda$, $\lambda > 1/2$, $0 \le \lambda_0 \le \lambda \le d+1$, $\lambda_0 < d' + \varrho$, we obtain

$$\|g - S_n g\|_{\lambda_0} \le c \cdot 2^{-n(\lambda-\lambda_0)}\|g\|_\lambda$$

(approximation property; it follows from [3, Theorem 5.2]).

2. If $d' + \varrho > r \ge t$, then for all $\psi \in Y_n$

$$\|\psi\|_r \le c \cdot 2^{n(r-t)}\|\psi\|_t$$

(inverse property; it follows from [3, Theorem 5.1]; in particular, $\|\psi\|_{\lambda_0} \le c \cdot 2^{n\cdot\lambda_0}\|\psi\|_0$ holds).

3. For $\psi \in Y_n$ we obtain

$$\|\psi\|_0 \le c \cdot \max_{k\in\mathbb{Z}_n} |\psi(t_k^n)|$$

(finite property). To prove it we use the stability assertion

$$\Big\|\sum_{k\in\mathbb{Z}_n} c_k\varphi_k^n\Big\|_0 \sim \Big(\sum_{k\in\mathbb{Z}_n} |c_k|^2\Big)^{1/2}$$

following from property (P2) (cf. [3]). For $\psi \in Y_n$ we have

$$\psi = S_n\psi = \sum_{k\in\mathbb{Z}_n} 2^{-n/2}\psi(t_k^n)\,\varphi_k^n$$

hence

$$\|\psi\|_0^2 = \Big\|\sum_{k\in\mathbb{Z}_n} 2^{-n/2}\psi(t_k^n)\,\varphi_k^n\Big\|_0^2 \le c\cdot 2^{-n}\sum_{k\in\mathbb{Z}_n}|\psi(t_k^n)|^2 \le c\cdot\max_{k\in\mathbb{Z}_n}|\psi(t_k^n)|^2.$$

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